1 Continuous-time interest rate risk hedging in the optimal portfolio choice

1.1 Introduction

Since Merton [1971] the optimal consumption-portfolio choice problem in the financial economics literature is formulated in the convenient continuous-time framework. Merton [1971] solved the problem in question by the application of stochastic dynamic programming. Merton’s [1971] solution shows that the demand for risky assets in the optimal portfolio consists of mean-variance and hedging components. The first component characterizes the optimal portfolio when the investment opportunity set is at most deterministic or the utility function is logarithmic, while the latter appears in a stochastic setting. In particular, stochastic interest rates, market prices of risk and inflation rates give rise to the hedging constituents. The mean-variance demand is a demand by a single period mean-variance maximizer. The hedging demand is a vehicle to hedge against unfavourable shifts in the investment opportunity set.

The stochastic dynamic programming has become a common way to solve the continuous-time consumption-portfolio choice problem. This method was applied among others by Sorensen [1999], Brennan and Xia [2000] and Munk et al. [2004]. Sorensen [1999] and Brennan and Xia [2000] solve the problem of maximizing expected constant relative risk aversion (CRRA) utility from terminal wealth. However, they differ in the model of interest rate dynamics. In Sorensen [1999] this dynamics is described by Vasicek [1977] model, while in Brennan and Xia [2000] by the Hull and White [1996] two-factor model. They show that the positions taken in stock and bond are ceteris paribus increasing functions of the expected excess returns and that the relevant hedge portfolio is the zero coupon bond with expiration at the investment horizon. For log-utility investors the hedging term vanishes. Stock allocations are determined solely by the myopic portfolio, because the hedge portfolio does not contain equity. As long as the bond has a maturity equal to the investment horizon, the optimal allocations to stock, bond and cash are independent of the horizon. The stock allocation is decreasing and the cash and bond allocations are increasing in the investor’s risk aversion. Munk et al. [2004] consider dynamic asset allocation under mean-reverting returns, stochastic interest rates and inflation uncertainty. They try to explain simultaneously the Samuelson [1963] and Canner et al. [1997] puzzles. Munk et al. [2004] consider an investor with the objective to maximize the expected CRRA utility from the real terminal wealth. The excess return on the stock index as well as the nominal interest rate dynamics are described by an Ornstein-Uhlenbeck process. So is the expected rate of inflation. In the solution to the problem Munk et al. [2004] obtain three hedging terms. The optimal hedge against changes in the expected equity excess returns is obtained by investing exclusively in stocks, which is due to the perfect negative correlation between the stock process and the excess return process. The optimal
hedge against changes in the interest rate is obtained by investing exclusively in the bond, which is perfectly negatively correlated with short interest rate. The inflation hedge involves both the stock and the bond.

The problem introduced by Merton [1971] with an additional non-negativity constraint on terminal wealth was solved by means of the martingale method by Cox and Huang [1989]. In contrast to the stochastic dynamic programming where the optimization is over the current consumption rate and the fractions of current wealth invested in the risky securities the approach of Cox and Huang [1989] is a two-step procedure. In the first step the distribution of optimal terminal wealth is found, while in the second stage the unique replicating portfolio is determined for which purpose the partial differential equation (PDE) needs to be solved. With the application of martingale techniques explicit portfolio rules were derived for instance by Bajeux-Besnainou et al. [2003] for the hyperbolic absolute risk aversion (HARA) utility over the terminal wealth and Vasicek [1977] interest rate dynamics, by Deelstra et al. [2000] for CRRA utility over the terminal wealth and Cox-Ingersoll-Ross (CIR) [1985] interest rate dynamics and by Munk and Sorensen [2004] for CRRA utility over the terminal wealth as well as consumption and the interest rates having Heath-Jarrow-Morton [1992] multifactor Gaussian dynamics. The hedge bond in the setting of Munk and Sorensen [2004] is the coupon bond with coupon rates equal to the certainty equivalents of optimally planned future consumption rates. The models of the financial market in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] account for the influence of stochastic interest rate on the stock prices by modelling stock returns as two-factor diffusions with one of the Brownian motions being perfectly correlated with interest rate volatility term. In both it is shown that the bond with maturity matching the investor horizon is the proper hedge bond. The models of Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] are subject of our interest in this chapter.

Though general formulas for wealth replicating portfolio exist (cf. Ocone and Karatzas [1991]) an explicit identification of this portfolio still proves to be cumbersome in more complex, and at the same time more realistic, models with more hedging terms. This is partly due to the expression of risky assets demand in the form of conditional expectations of random variables involving (also stochastic) integrals of Malliavin derivative of state variables what is shown in Ocone and Karatzas [1991]. In the recent paper Detemple et al. [2003] demonstrate that these derivatives follow diffusion processes. Consequently, they propose to use Monte Carlo simulation to approximate the expectations in question and consequently to simulate the optimal portfolio fractions of risky assets.

In this chapter we consider first a CRRA investor. We position him in two models of financial market, respectively as in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000]. Hence, we know the optimal policy an investor should follow and its decomposition into the mean-variance and hedging portfolios if he rebalances them continuously. Moreover, the selection of financial markets as in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] has the advantage of stochastic interest rate being (apart from the risky assets) the only state
variable. This is because the market prices of risk are constant in the former
and at most a function of interest rate in the latter. As the sole source of risk
stems from the interest rate dynamics, the interest rate risk hedging component
is eventually the only hedging constituent that shows up in the risky assets
demand. This in turn eases the simulation of optimal portfolios in the Monte
Carlo experiment. Subsequently, we consider a HARA investor in the model of
the financial market as in Bajeux-Besnainou et al. [2003] and Deelstra et al.
[2000]. While in the former model the literature provides us with the answer
what the optimal portfolio strategy should be, in the latter model the analytical
formulas are not available to the best of our knowledge even though the dynamics
of interest rates is not too complex yet.

Our initial focus on CRRA investor serves well one of the goals of this chapter
which is to introduce the concept of interest rate risk hedging in the continuous-
time setting. The convenience of CRRA utility is connected with two aspects.
First, by using CRRA utility the optimal hedging portfolio policy is not blurred
by the liability hedging component, which appears with HARA utility. Sec-
ond, as the optimal portfolio policy in the chosen models of financial market is
known for CRRA utility we can investigate the convergence of CRRA portfolio
simulated according to the guidelines of Detemple et al. [2003] and its con-
stituents to the analytical equivalents given in the literature. Our second goal
is to present how one can obtain analytical formulas for the optimal portfolios
as in Bajeux-Besnainou et al. [2003] using the Malliavin calculus approach of
Detemple et al. [2003] and Ocone and Karatzas [1991]. Our third objective is
to simulate HARA portfolios with the hedging terms distinguished when the
interest rate has the CIR [1985] dynamics and to show how the hedging terms
react to the changes in the ratio of initial wealth to the subsistence level.

The chapter is organized as follows. In section 2.2 we describe the martingale
approach to the valuation of contingent claims and to the optimal wealth
choice. In section 2.3 we describe the partial differential equation approach to
deriving the optimal portfolios and introduce the approach of Detemple et al.
[2003] to simulating optimal portfolios. In section 2.4 we describe two models
of financial market, as in Bajeux-Besnainou et al. [2003] and Deelstra et al.
[2000]. Section 2.5 presents the solution to the optimal portfolio choice problem
in the considered models of financial market derived by Bajeux-Besnainou et al.
[2003] and Deelstra et al. [2000]. It also explains in the introduced models the
details of simulating optimal portfolios according to Detemple et al. [2003]. In
this section we also show how the results of Bajeux-Besnainou et al. [2003] can
be obtained from the Malliavin derivative approach to calculating the optimal
portfolios. Section 2.6 presents the assumptions of a simulation experiment.
In section 2.7 simulated CRRA portfolio rules are reported and their conver-
gence to the continuous-time limit illustrated. Also HARA simulated portfoli-
os and the relation between hedging and initial wealth-to-subsistence-level ratio is
presented in this section. Section 2.11 concludes the chapter.
1.2 Martingale approach

1.2.1 Valuation of contingent claims

The exposition in this section is based on Duffie [2001], Pelsser [2000], Björk [1998], Baxter and Rennie [1996], Sundaram [1997] and Bajeux-Besnainou and Portait [1997]. Let us specify the model of the continuous trading economy with a trading taking place between time $0$ and $T$: We fix a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega$ is the sample space, $\mathcal{F}$ is the $\sigma$-algebra on $\Omega$ and $P$ is the probability measure on $(\Omega, \mathcal{F})$. The probability space is constructed so that there exist standard Brownian motions. The uncertainty is resolved over $[0,T]$ according to the filtration $\{\mathcal{F}_t\}$.

There are $N + 1$ marketed assets. Let their prices $S = (S^0, S^1, ..., S^N)$ be modelled as Ito processes

$$dS^i(t) = \mu^i(t)S^i(t)dt + \sigma^i(t)S^i(t)dz(t), \quad (2.1)$$

$i = 0, 1, ..., N$. Security $0$ is a risk-free asset with $\mu^0(t) = r(t)$ and $\sigma^0(t) = 0$, where $r(t)$ is the short rate process. For instance, when $N = 2$ we may think of $S^1(t)$ as the price process of a bond and of $S^2(t)$ as the price process of a stock. Drifts $\mu^i(t)$ and the vectors of volatilities $\sigma^i(t)$ being the $i$-th row of volatility matrix $\sigma(t)$ are bounded adapted processes. We assume that matrix $\sigma(t)$ is invertible.

We assume that $dz(t)$ is an increment of $N$-dimensional Brownian motion. As the number of random sources equals the number of underlying traded assets excluding the risk free asset the model is complete and free of the arbitrage opportunities. An arbitrage opportunity is a self-financing trading strategy which has strictly negative initial costs and with probability 1 has a non-negative value at time $T$. The economy is called complete if all derivative assets are attainable in it. A derivative security is defined via its uncertain payoff at time $T$ denoted by $X(T)$ and being an $\mathcal{F}_T$-measurable random variable with the property that the expectation of payoff for these derivatives is well-defined. The derivative is said to be attainable if we can find a self-financing trading strategy such that the value of the portfolio at time $T$ obtained via the following of this strategy equals $X(T)$ with probability 1. A trading strategy is a predictable $N$-dimensional stochastic process of the holdings in each of the $N$ assets at time $t$. The self-financing trading strategy is then called a replicating strategy. A self-financing trading strategy is a strategy that neither requires nor generates funds between time 0 and time $T$.

Any asset which has strictly positive prices for all $t \in [0,T]$ is called a numeraire. We choose the riskless asset to be the numeraire. We write the deflated processes $\hat{S}^i(t)$ as

$$\hat{S}^i(t) = S^i(t) \exp \left( - \int_0^t r(u)du \right).$$
Numeraire invariance principle says that a trading strategy is self-financing with respect to $S$ if and only if it is self-financing with respect to $\hat{S}$, $\hat{S} = (\hat{S}_0, \hat{S}_1, ..., \hat{S}_N$). A trading strategy is an arbitrage with respect to $S$ if and only if it is an arbitrage with respect to the deflated price process $\hat{S}$.

Conditions for the financial market to be complete and arbitrage free can be also formulated in terms of the equivalent martingale measure $Q$. A continuous economy is free of arbitrage opportunities and complete if there exists a unique equivalent martingale measure. A probability measure $Q$ on $(\Omega, \mathcal{F})$ is said to be equivalent to $P$, provided for any event $A$, we have $Q(A) > 0$ if and only if $P(A) > 0$. An equivalent probability measure is an equivalent martingale measure for the discounted price processes $\hat{S}_i(t)$, $i = 1, 2, ..., N$, if $\hat{S}_i(t)$ are martingales with respect to $Q$ and if the Radon-Nikodym derivative $\frac{dQ}{dP}$ has finite variance.

Let $\lambda(t)$ defined as
\begin{equation}
\lambda(t) = \sigma^{-1}(t) (\mu(t) - r(t) \mathbf{1})
\end{equation}
be the vector of market prices of risk, where $\mathbf{1}$ is a vector of $N$ ones. For any $\lambda(t)$ such that
\[ \int_0^t \lambda^T(s)\lambda(s)ds < \infty, \]
with probability 1, the Radon-Nikodym derivative is given by
\[ \frac{dQ}{dP} = \zeta(T) = \exp \left( -\int_0^T \lambda^T(s)dz(s) - \frac{1}{2} \int_0^T \lambda^T(s)\lambda(s)ds \right). \]
Under the measure $Q$ the process
\begin{equation}
z^Q(s) = z(s) + \int_0^t \lambda(s)ds
\end{equation}
is a $Q$–Brownian motion. The above theorem is known as the Girsanov theorem.

It is not difficult to show that $S^i(t)$ have drifts equal to the short term interest rate under the measure $Q$. From formula (2.2)
\[ \mu^i(t) = \sigma^i(t)\lambda(t) + r(t). \]
Substituting into formula (2.1) we obtain the price processes under $P$
\begin{equation}
dS^i(t) = (\sigma^i(t)\lambda(t) + r(t)) S^i(t)dt + \sigma^i(t)S^i(t)dz(t).
\end{equation}
From (2.3)
\[ dz(s) = dz^Q(s) - \lambda(t)dt. \]
Substituting (2.5) into (2.4) we end up with
\[ dS^i(t) = r(t)S^i(t)dt + \sigma^i(t)S^i(t)dz^Q(s). \]
Hence, discounted price processes are martingales under $Q$ and the measure $Q$ defined in the Girsanov theorem is the equivalent martingale measure.

For a probability measure $Q$ equivalent to $P$, the density process $\zeta(t)$ for $Q$ is a martingale defined by

$$\zeta(t) = E_t \left( \frac{dQ}{dP} \right) = \exp \left( - \int_0^t \lambda^T(s)dz(s) - \frac{1}{2} \int_0^t \lambda^T(s)\lambda(s)ds \right),$$

$t \in [0,T]$. When the market price of risk is bounded the stochastic discount factor also known as the pricing kernel or state price density is defined by

$$\zeta(t) = \exp \left( - \int_0^t r(u)du \right) \zeta(t). \quad (2.6)$$

The reciprocal of $\zeta(t)$ is shown for instance in Merton (1990) to be the value of growth-optimal portfolio when all dividends are reinvested. The growth-optimal portfolio is the portfolio that maximizes the log return on wealth and it is the optimal portfolio for an investor with log utility of terminal wealth.

As the discounted marketed assets are martingales under the measure $Q$ then also a discounted derivative asset if attainable is a martingale under $Q$ what we write

$$E_t^Q \left( \exp \left( - \int_0^T r(u)du \right) X(T) \right) = \left( - \int_0^t r(u)du \right) X(t).$$

Changing the measure from $Q$ to $P$ we have

$$E_t \left( \exp \left( - \int_0^T r(u)du \right) X(T)\zeta(s) \right) = \left( - \int_0^t r(u)du \right) X(t)\zeta(t). \quad (2.7)$$

From (2.7) we have

$$X(t) = \frac{E_t (\zeta(T)X(T))}{\zeta(t)},$$

for any times $t$ and $T > t$ and any $\mathcal{F}_T$-measurable random variable $X(T)$ such that $E^Q (|X(T)|) < \infty$, which is the price of the derivative at time $t$.

### 1.2.2 Optimal wealth choice

In the martingale approach to the optimal portfolio choice the following steps are involved. First, the dynamic problem from stochastic control approach is transformed into the static problem. Second, using the Lagrange multiplier rule the problem is solved for optimally invested wealth. Third, an optimal replicating strategy is found. In this section we deal with the first two steps.
Let us start with the stochastic control approach to the optimal investment problem in continuous time. The problem in question which is to maximize the expected utility from terminal under the budget restriction can be written as

\[
\max_{(X_T, \pi)} E_t u(X(T))
\]

\[\text{s.t. } \frac{dX(t)}{X(t)} = \left( \pi^T(t) \sigma(t) \lambda(t) + r(t) \right) dt + \pi^T(t) \sigma(t) dz(t), \quad (2.8)\]

where \( \pi(t) \) is an adapted process \( \pi(t) = (\pi(t)^1, ..., \pi(t)^N)^T \) defining fractions of total wealth held in the risky securities and \( X(t) \) is the wealth process. The utility function \( u(*) \) is strictly concave, increasing and differentiable on \((0, 1)\) and satisfies the Inada conditions which are \( \inf_{X_T} u_{X_T}(X(T)) = 0 \) and \( \sup_{X_T} u_{X_T}(X(T)) = \infty \). The dynamic budget constraint can be interpreted as the total return on an asset whose price is \( X(t) \).

In such a case the discounted wealth process should be a martingale under the measure \( Q \). We will verify that indeed this is the case.

Applying Ito’s lemma to the process (2.1) we obtain the dynamics of state price density as follows

\[
d\zeta(t) = \zeta(t) \left[ -r(t) dt - \lambda^T(t) dz(t) \right].
\]

Applying the product rule to \( d\zeta(t)X(t) \) we have

\[
d\zeta(t)X(t) = \zeta(t)dX(t) + X(t)d\zeta(t) + d\zeta(t)dX(t) = \zeta(t)X(t) \left( \pi(t) \sigma(t) - \lambda^T(t) \right) dz(t).
\]

Hence, integrating both sides leads to

\[
\zeta(T)X(T) = \zeta(t)X(t) + \int_t^T \zeta(u)X(u) \left( \pi(u) \sigma(u) - \lambda^T(u) \right) dz(u).
\]

Taking expectations of both sides we end up with the constraint

\[
E_t [\zeta(T)X(T)] = \zeta(t)X(t).
\]

Hence, in the first step we can rewrite the stochastic control problem with dynamic budget constraint as the optimization problem with static budget restrictions in the form of martingale constraints as follows

\[
\max_{X_T} E_t u(X(T))
\]

\[\text{s.t. } E_t [\zeta(T)X(T)] = \zeta(t)X(t), \quad (2.9)\]

where the optimization is only upon \( X(T) \). Cox and Huang [1989] show that given a terminal wealth \( X^*(T) \) and some initial wealth \( X^*(t) \) there exists a trading strategy \( \pi^*(t) \) such that \( X^*(T) \) and \( \pi^*(t) \) solve the stochastic control
problem if and only if \(X^*(T)\) solves the above static problem. In order to solve for \(X^*(T)\) the Lagrangian multiplier method is used. The wealth \(X^*(T)\) solves the problem \((2.9)\) if and only if there is a Lagrange multiplier \(\kappa^* > 0\) such that \(X^*(T)\) solves the unconstrained problem
\[
\max_{X_T} E_t [u(X(T)) - \kappa E_t [\zeta(T)X(T) - \zeta(t)X(t)]]
\]
and satisfies the budget restriction. We write the Lagrangian as
\[
L = E_t [u(X(T)) - \kappa (\zeta(T)X(T) - \zeta(t)X(t))] = \int_{\Omega} [u(X(T)) - \kappa (\zeta(T)X(T) - \zeta(t)X(t))] dP(\omega).
\]
The first-order condition for optimality of \(X^*(T)\) is
\[
\frac{\delta L}{\delta X(T, \omega)} = u_X(X(T)) - \kappa \zeta(T) = 0,
\]
that is
\[
X^*(T) = I(\kappa \zeta(T)),
\]
where \(I(\bullet) = u^{-1}_X(\bullet)\). Having \(X^*(T)\) we may substitute it back into the budget constraint and solve for the correct Lagrange multiplier \(\kappa^*\). With that multiplier we end up with the optimally invested terminal wealth.

### 1.3 Optimal portfolio choice

#### 1.3.1 PDE approach

However, the terminal wealth is not all. The next step is to determine the process \(X(t)\) for every \(0 \leq t \leq T\) and the wealth replicating strategy. In general it is not easy to find the replicating strategies. The approach based on the Malliavin calculus presented in the papers of Karatzas and Ocone [1991] and in Detemple et al. [2003] is discussed in the next section. In this section we demonstrate the approach based on the partial differential equation (PDE). The exposition is mainly based on Campbell and Viceira [2002].

Let us denote by \(Z(t)\) the reciprocal of \(\zeta(t)\). Given the Markovian structure of the dynamics for \(Z(t)\) and any state variable \(V(t)\) the expectation from the budget constraint will be some function \(F\) of the current value of \(Z(t)\) and if the process for \(Z(t)\) depends on the state variable \(V(t)\) it will also be a function of the current value of \(V(t)\) what we write
\[
X(t) = F(Z(t), V(t), t).
\]

The wealth discounted with the stochastic discount factor has the martingale property
\[
E_t [d(\zeta(t)X(t))] = E_t [d(\zeta(t)F(t))] = 0.
\]
The above expectation implies a second-order partial differential equation for optimally invested wealth. To compute this expectation we need first to compute \( d(\zeta(t)F(t)) \). By the product rule
\[
d(\zeta(t)F(t)) = \zeta(t) dF(t) + F(t) d\zeta(t) + d\zeta(t) dF(t).
\]
(2.10)

Using Ito’s lemma the dynamics of \( dF(t) \) is given by
\[
dF(t) = \mu_V(V,t) dt + \sigma_V^T(V,t) dz(t),
\]
where \( \sigma_V(V,t) \) is \( N \times 1 \) vector and \( \mu_V(V,t) \) is a scalar. From Ito’s lemma it can be shown that \( dZ \) follows
\[
dZ(t) = Z(t) \left[ (r(V,t) + \lambda(V,t)^T \lambda(V,t)) dt + \lambda^T(V,t) dz(t) \right].
\]

From the substitution of \( dF(t) \) and \( d\zeta(t) \) into equation (2.10) we recognize that the condition that the drift term of \( d(\zeta(t)F(t)) \) equals zero writes as

\[
F_Z \mu_V - \lambda^T \sigma_V + F_t + \frac{1}{2} F_{ZZZ} \lambda^2 + \frac{1}{2} F_{ZZ} \lambda + \frac{1}{2} F_{VV} \sigma_V^T \sigma_V - F_Z F_V Z \lambda^T \sigma_V - F r = 0.
\]

For clearer exposition we omitted the dependence on \( V \) and \( t \). Once we solve the above PDE we obtain \( X(t) \). Then we can also solve for the optimal portfolio choice. To solve for optimal portfolio we simply equate the diffusion terms of the intertemporal budget constraint in (2.8) and the PDE describing the dynamics of optimally invested wealth (2.11), since both must be the same along the optimal path
\[
\pi(t) \sigma(t) X(t) dz(t) = F_Z \lambda^T(V,t) dz(t) + F_V \sigma_V^T(V,t) dz(t).
\]

Hence,
\[
\pi(t) = (\sigma^T(t))^{-1} \frac{F_Z \lambda(V,t)}{X(t)} + (\sigma^T(t))^{-1} \frac{F_V \sigma_V(V,t)}{X(t)}.
\]

In the above formula we can identify the first component as the myopic term in the optimal portfolio rule and the second component as the hedging term. Myopic term describes the portfolio which is optimal when the investor is a single-period mean-variance maximizer, when the investment opportunity set is at most deterministic or when the investor such as the log-utility maximizer does not hedge against changes in the investment opportunity set. Hedging constituent appears due to the long-term investment horizon, stochastic investment opportunity set and non-logarithmic utility.
1.3.2 Malliavin calculus

Solving the partial differential equation from the previous section is not always possible. Consequently, an explicit optimal trading strategy is not always found either. Instead of solving the partial differential equation, Detemple et al. [2003] in a recent paper propose to simulate the solution to the portfolio choice problem. The simulation is possible because the demand for risky assets is expressed as expected values of expressions detailed in Ocone and Karatzas [1991]. Theorem 1 in Detemple et al. [2003] says that the fractions invested in the risky assets $\pi(t)$ are given by

$$
\pi(t) = (\sigma^T)^{-1} \left( \frac{\lambda}{R(X_t)} c(t, \lambda, r) - a(t, \lambda, r) - b(t, \lambda, r) \right),
$$

where

$$
a(\cdot)^T \equiv \mathbb{E}_t \left( \zeta_{t,T} \frac{X_T}{X_t} \left( 1 - R(X_T)^{-1} \right) 1_{X_T > 0} \int_t^T D_t r_s ds \right), \quad (2.13)
$$

$$
b(\cdot)^T \equiv \mathbb{E}_t \left( \zeta_{t,T} \frac{X_T}{X_t} \left( 1 - R(X_T)^{-1} \right) 1_{X_T > 0} \int_t^T (dz_s + \lambda_s ds)^T D_t \lambda_s \right), \quad (2.14)
$$

$$
c(\cdot) \equiv \mathbb{E}_t \left( \zeta_{t,T} \frac{X_T}{X_t} \frac{R(X_t)}{R(X_T)} 1_{X_T > 0} \right). \quad (2.15)
$$

In the above formulas $1_{X_T > 0}$ is the indicator of the event $X_T > 0$, $R(x)$ is the relative risk aversion coefficient defined as

$$
R(x) = -\frac{u_{xx}(x)x}{u_x(x)},
$$

while $\zeta_{t,T} = \frac{\zeta_T}{\zeta_t}$.

Let us generalize the framework and let $V(t)$ denote the $N$-dimensional vector of state variables. Then $\mu^V$ becomes a vector and $\sigma^V$ becomes a matrix. Let $D_t r_s$ and $D_t \lambda_s$ stand for the vector and matrix respectively of Malliavin derivatives of interest rate and market prices of risk. They are given by

$$
D_t \lambda_s = \partial_2 \lambda(s, V(s)) D_t V(s),
$$

$$
D_t r_s = \partial_2 r(s, V(s)) D_t V(s),
$$

where the Malliavin derivative of state variable $D_t V(s) = (D_t r_V(s), ..., D_{Nt} r_V(s))$ solves the linear stochastic differential equation

$$
d \left( D_{kt} V(s) \right) = \partial_2 \mu^V(s, V(s)) D_{kt} V(s) ds + \left( \sum_{j=1}^N \partial_2 \sigma^V_j(s, V(s)) dz_j(s) \right) D_{kt} V(s),
$$

subject to the boundary condition $\lim_{s \to t} D_{kt} V(s) = \sigma^V_k(s, V(s))$. In the above formula $\sigma^V_j(s, V(s))$ is the $j$th column of the matrix $\sigma^V(s, V(s))$ and $\partial_2 \sigma^V_j(s, V(s))$
is the gradient with respect to \( V(s) \) of \( \sigma_j^V(s, V(s)) \), \( j = 1, ..., N \). The theorem shows that the Malliavin derivatives of state variables satisfy diffusion processes what implies that the simulation methods can be used to calculate the portfolio shares. The Malliavin derivatives capture the impact of an innovation in the Brownian motion \( z(t) \) at time \( t \) on the state variable \( V(s) \) at time \( s \).

The first component of the portfolio (2.12) is a mean-variance term while the next two are intertemporal hedging terms. In this decomposition of replicating portfolio \( \pi(t) \equiv - (\sigma^T)^{-1} a(\cdot) \) is termed the interest rate risk hedging component and \( \pi^{MPR}(t) \equiv - (\sigma^T)^{-1} b(\cdot) \) is called the market price of risk hedging constituent. These terms measure respectively the interest rate and market price of risk sensitivity to the underlying Brownian motions \( z(t) \). The Malliavin derivatives differ from zero when the investment opportunity set is stochastic and then the hedging matters for the optimal asset allocation.

When the utility is the HARA function with the subsistence level \( \bar{X} \) it has the form

\[
u(x) = \frac{\gamma}{1-\gamma} \left( \frac{x - \bar{X}}{\gamma} \right)^{1-\gamma}.
\]

The relative risk risk aversion coefficient for the HARA utility is equal to

\[
R(x) = \frac{\gamma x}{x - \bar{X}}.
\]

Optimal terminal wealth for HARA with the subsistence level \( \bar{X} \) is given by the formula

\[
X(T) = \xi_{t,T}^{-\frac{1}{\gamma}} X_t - \bar{X} \hat{E}_t \left( \xi_{t,T} \right) + \bar{X}.
\]

The above formulas can be used in the numerical procedure to compute the mean-variance and hedging portfolios. The key is to simulate the \( M \) trajectories of \( \zeta_{t,s}, H_r^t, \) and \( H_{\lambda}^t \) which according to Ito’s lemma follow

\[
d\zeta_{t,s} = -\zeta_{t,s} \left( r(s, V_s) ds + \lambda^T(s, V_s) dz(s) \right),
\]

\[
dH_r^t = \partial_2 r(s, V_s) D_t V_s ds,
\]

\[
dH_{\lambda}^t = (dz(s) + \lambda(s, V_s) ds) \partial_2 \lambda(s, V_s) D_t V_s ds,
\]

where

\[
H_r^t = \int_t^s D_t r_v dv,
\]

\[
H_{\lambda}^t = \int_t^s (dz(v) + \lambda(v, V_v) dv) \partial_2 \lambda_s D_t v.
\]

The simulation can be done using an Euler scheme that discretizes the time interval in \( N \) points. We can use the set of \( M \) estimates \( \zeta^{N,1}_{t,s}, H^{r,N,1}_{t,s}, H^{\lambda,N,1}_{t,s} \)
and $X_T^{N,i}$, $i = 1, 2, ..., M$, of $\zeta_{t,s}$, $H_{t,s}^T$, $H_{t,s}^\lambda$ and $X_T$ to provide us with the estimates of the functions $a(\cdot)^T$, $b(\cdot)^T$ and $c(\cdot)$ in the hedges, which for HARA utility with the subsistence level are equal to

$$a(\cdot)^T = \frac{1}{M} \sum_{i=1}^{M} \zeta_{t,T}^N X_T^{N,i} \left( 1 - R(X_T^{N,i})^{-1} \right) H_{t,s}^T,$$

$$b(\cdot)^T = \frac{1}{M} \sum_{i=1}^{M} \zeta_{t,T}^N X_T^{N,i} \left( 1 - R(X_T^{N,i})^{-1} \right) \int_t^T (dz_s + \lambda_s ds)^T H_{t,s}^\lambda,$$

$$c(\cdot) = \frac{1}{M} \sum_{i=1}^{M} \zeta_{t,T}^N X_T^{N,i} \frac{R(X_t)}{R(X_T^{N,i})}.$$

When the utility is the CRRA function given by

$$u(x) = \frac{1}{\gamma} x^\gamma$$

then the Arrow-Pratt measure of relative risk aversion is constant and equal to $R(x) = 1 - \gamma$. As the non-negativity constraint is not binding for power utility we have $1_{x_T > 0} = 1$. Accounting additionally for the budget constraint results in $c(\cdot) = 1$. The expressions for $a(\cdot)^T$ and $b(\cdot)^T$ then simplify to

$$a(\cdot)^T = \frac{\gamma}{\gamma - 1} E_t \left( \zeta_{t,T}^T \frac{X_T}{X_t} \int_t^T \left( dz_s + \lambda_s ds \right)^T D_t \lambda_s \right),$$

$$b(\cdot)^T = \frac{\gamma}{\gamma - 1} E_t \left( \zeta_{t,T}^T \frac{X_T}{X_t} \int_t^T \left( dz_s + \lambda_s ds \right)^T D_t \lambda_s \right).$$

For CRRA utility the terminal wealth is given by

$$X(T) = \frac{\zeta_{t,T}^T X(t)}{E_t \left( \zeta_{T}^T \right)}.$$

Substituting into (2.19)-(2.20) and simplifying we end up with

$$a(\cdot)^T = \frac{\gamma}{\gamma - 1} \frac{E_t \left( \zeta_{t,T}^T \int_t^T \left( dz_s + \lambda_s ds \right)^T D_t \lambda_s \right)}{E_t \left( \zeta_{T}^T \right)},$$

$$b(\cdot)^T = \frac{\gamma}{\gamma - 1} \frac{E_t \left( \zeta_{t,T}^T \int_t^T \left( dz_s + \lambda_s ds \right)^T D_t \lambda_s \right)}{E_t \left( \zeta_{T}^T \right)}.$$

In case of CRRA utility the simulation of optimal portfolios can also be done using an Euler scheme that discretizes the time interval in $N$ points. We can
use the set of $M$ estimates $\hat{\zeta}_{t,s}^{N,i}$, $H_{t,s}^{r,N,i}$ and $H_{t,s}^{\lambda,N,i}$, $i = 1,2,\ldots,M$, to construct the estimates $\left(\hat{\zeta}_{t,s}^{N,i}\right)^{\frac{1}{\gamma}} H_{t,s}^{r,N,i}$ and $\left(\hat{\zeta}_{t,s}^{N,i}\right)^{\frac{1}{\gamma}} H_{t,s}^{\lambda,N,i}$ of $\left(\hat{\zeta}_{t,s}\right)^{\frac{1}{\gamma}} H_{t,s}^{r}$ and $\left(\hat{\zeta}_{t,s}\right)^{\frac{1}{\gamma}} H_{t,s}^{\lambda}$. Averaging over $M$ provides estimates of the functions $a(\cdot)^T$ and $b(\cdot)^T$ in the hedges, which for CRRA utility are equal to

\[
\hat{a}(\cdot)^T = \frac{\gamma}{\gamma - 1} \sum_{i=1}^{M} \left(\hat{\zeta}_{t,s}^{N,i}\right)^{\frac{1}{\gamma}} H_{t,s}^{r,N,i},
\]

\[
\hat{b}(\cdot)^T = \frac{\gamma}{\gamma - 1} \sum_{i=1}^{M} \left(\hat{\zeta}_{t,s}^{N,i}\right)^{\frac{1}{\gamma}} H_{t,s}^{\lambda,N,i}.
\]

### 1.4 Financial market models

We consider a financial market which is arbitrage-free, complete and frictionless. Let $z(t) = [z(t), z_r(t)]^T$, $t \geq 0$, denote standard two-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Evolution of the interest rate is described by

\[
\begin{align*}
&dr(t) = a(b - r(t))dt - \sigma_r (r(t))^\eta dz_r. & (2.21)
\end{align*}
\]

We focus on two models for the financial market. First, we choose $a, b, \sigma_r > 0$ and $\eta = 0$ which results in the Vasicek [1977] model and second we select $a, b, \sigma_r > 0$ and $\eta = \frac{1}{2}$ which yields the CIR [1985] model.

There are three assets available in the market, a stock index with price at time $t$ equal to $S(t)$, a bond maturing at $T$ with price $B_T(t)$, and a money market account, $B(t)$. Following Bajeux-Besnainou et al. [2003] ($\eta = 0$) and Deelstra et al. [2000] ($\eta = 0.5$) we specify their price processes as

\[
\begin{align*}
&dS(t) = S(t) \left[r(t)dt + \sigma_1 (dz + \lambda dt) + \sigma_2 (r(t))^\eta (dz_r + \lambda_r (r(t))^\eta dt)\right], \\
&dB_T(t) = B_T(t) \left[r(t)dt + \sigma_T(t) (dz_r + \lambda_r (r(t))^\eta dt)\right], \\
&dB(t) = B(t) [r(t)dt].
\end{align*}
\]

When $\eta = 0$ the instantaneous volatility of the $T$-maturity bond is given by

\[
\sigma_T(t) = \sigma_r a^{-1} (1 - \exp(-a(T - t))), & (2.22)
\]

while when $\eta = \frac{1}{2}$ this volatility equals

\[
\sigma_T(t) = \sigma_r h(T - t) \sqrt{r(t)}. & (2.23)
\]

---

1. Following Bajeux-Besnainou et al. [2003] we use $-\sigma_r$ instead of $+\sigma_r$. As noted therein this assures positive volatilities of bonds and typically observed negative correlation between interest rates and bond as well as stock prices.
with
\[ h(s) = 2(\exp(\delta s) - 1) [\delta - (a - \sigma_r \lambda_r) + \exp(\delta s) (\delta + a - \sigma_r \lambda_r)]^{-1}, \]
\[ \delta = \sqrt{(a - \sigma_r \lambda_r)^2 + 2\sigma_\gamma^2}. \]

In the above formulas \( \lambda \) and \( \lambda_r \) (\( r(t) \)) denote stock and bond market prices of risk respectively, while \( \sigma_1 \) and \( \sigma_2 \) stand for stock index volatilities. Parameters \( \lambda, \lambda_r, \sigma_1 \) and \( \sigma_2 \) are positive constants. Let \( \sigma \) and \( \lambda \) stand for the volatility matrix and the vector of market prices of risk respectively

\[ \sigma = \begin{bmatrix} \sigma_1 & \sigma_2 (r(t)\gamma) \\ 0 & \sigma_T (t) \end{bmatrix}, \]
\[ \lambda = \begin{bmatrix} \lambda \\ \lambda_r (r(t)\gamma) \end{bmatrix}. \]

For the needs of portfolio simulation referred to in more detail in the previous section we discretize the continuous-time processes using the Euler scheme as follows: \( dt = \frac{1}{n}, dz \sim \mathcal{N}(0, \frac{1}{n}), dz_r \sim \mathcal{N}(0, \frac{1}{n}). \)

### 1.5 Formulas for optimal portfolios

#### 1.5.1 Analytical

Formulas for the proportion of stock and \( T \)-bond in the portfolio replicating optimal final wealth for an investor with the objective to maximize expected CRRA utility from terminal wealth in the model of Bajeux-Besnainou et al. [2003] are given therein as

\[ \pi(t) = \begin{bmatrix} \pi^1(t) \\ \pi^2(t) \end{bmatrix} = \begin{bmatrix} \lambda \sigma_1 (1-\gamma) \sigma_1 + \lambda \sigma_2 (1-\gamma) \sigma_2 (r(t)\gamma) \\ (\sigma_T (t) + \lambda) \sigma_1 (1-\gamma) - \gamma \end{bmatrix}, \]

where \( \pi^1(t) \) denotes the fraction invested in stock and \( \pi^2(t) \) is the fraction invested in bond. In the above formulas mean-variance demands, \( \pi^{MV}(t) \), are calculated from

\[ \pi^{MV}(t) = \frac{(\sigma^T)^{t-1} \lambda}{1 - \gamma}. \]

It is straightforward to obtain the mean-variance demands as

\[ \pi^{MV}(t) = \begin{bmatrix} \pi^{1,MV}(t) \\ \pi^{2,MV}(t) \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{(1-\gamma)\sigma_1} \\ \frac{\lambda \sigma_2 (1-\gamma) \sigma_2 (r(t)\gamma)}{(1-\gamma)\sigma_1 \sigma_T (t)} \end{bmatrix}. \]

Consequently, the hedging demands are given by

\[ \pi^{H}(t) = \begin{bmatrix} \pi^{1,H}(t) \\ \pi^{2,H}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{1 - \gamma} \end{bmatrix}. \]
Because in the model in question both market prices of risk are constant, hedging is needed exclusively against the interest rate risk and is performed solely by the bond.

Formulas for the proportion of stock and $T$-bond in the portfolio replicating wealth optimal for CRRA investor in the model of financial market as described in Deelstra et al. [2000] are given therein as

$$\pi(t) = \begin{bmatrix} \pi^1(t) \\ \pi^2(t) \end{bmatrix} = \begin{bmatrix} \frac{\lambda r \sigma_1 - \lambda \sigma_2}{(1-\gamma)\sigma_1 \sigma_h (T-t)} + k \left( T - t, \frac{\gamma}{1-\gamma} \right) h^{-1} (T-t) \\ \frac{(1-\gamma)\sigma_1}{\lambda r \sigma_1 - \lambda \sigma_2} \end{bmatrix},$$

where

$$k (s, c) = -c \frac{(\exp(as) - 1) (2 + \lambda^2 (1 + c))}{\alpha - a - c \lambda r \sigma_r + (\alpha + a + c \lambda r \sigma_r) \exp(\alpha s)},$$

with

$$\alpha = \sqrt{a^2 + 2 \sigma_r^2 \mu}$$

and

$$\mu = -c \left( 1 + \frac{\lambda^2}{2} - \frac{\lambda r a}{\sigma_r} \right).$$

We calculate the mean-variance demands as

$$\pi^{MV}(t) = \begin{bmatrix} \pi^{1,MV}(t) \\ \pi^{2,MV}(t) \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{(1-\gamma)\sigma_1 \sigma_h (T-t)} \\ \frac{(1-\gamma)\sigma_1}{\lambda r \sigma_1 - \lambda \sigma_2} \end{bmatrix},$$

and consequently the hedging demands are

$$\pi^H(t) = \begin{bmatrix} \pi^{1,H}(t) \\ \pi^{2,H}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{k(T-t, \frac{\gamma}{1-\gamma})}{h(T-t)} \end{bmatrix}.$$

In the considered model the stock market price of risk is constant and thus no hedging against it is required. In turn, bond market price of risk given by $\lambda r \sqrt{r(t)}$ is a function of interest rate and as such the hedging against bond market price of risk reduces in fact to the hedging of interest rate risk. Though in both models of the financial market stock as well as bond returns are correlated with changes in interest rate, hedging against these changes is done solely by the asset whose volatility term is perfectly correlated with the increment of Brownian motion driving the interest rate. It results in no stock entering the hedging portfolio.

Formulas for the proportion of stock and $T$-bond in the portfolio replicating optimal final wealth for an investor with objective to maximize expected HARA utility from terminal wealth in the model of Bajeux-Besnainou et al. [2003] are given therein as

$$\pi(t) = \begin{bmatrix} \pi^1(t) \\ \pi^2(t) \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\gamma \sigma_1} \left( 1 - \frac{B_{\gamma}(t) \hat{X}}{\hat{X}(t)} \right) + \frac{1}{\gamma} + \frac{B_{\gamma}(t) \hat{X}}{\gamma \hat{X}(t)} \\ \frac{\lambda r \sigma_1 - \lambda \sigma_2}{\gamma \sigma_1 \sigma_r (t)} \left( 1 - \frac{B_{\gamma}(t) \hat{X}}{\hat{X}(t)} \right) \left( 1 - \frac{\gamma}{\gamma} \right) + \frac{B_{\gamma}(t) \hat{X}}{\gamma \hat{X}(t)} \end{bmatrix}. $$
In the above formulas mean-variance demands, \( \pi^{MV}(t) \), are calculated from
\[
\pi^{MV}(t) = (\sigma^T)^{-1} \frac{\lambda}{R(X_t)} c(t, \lambda, r).
\]

It is straightforward to obtain the mean-variance demands as
\[
\pi^{MV}(t) = \begin{bmatrix} \pi^{1,MV}(t) \\ \pi^{2,MV}(t) \end{bmatrix} = \begin{bmatrix} \lambda \sigma_1 \frac{1 - B(t) \tilde{X}}{X(t)} \\ \frac{\lambda \sigma_1 \lambda \sigma_2}{\gamma \sigma_1 \sigma_2 (t)} \left(1 - \frac{B(t) \tilde{X}}{X(t)} \right) \end{bmatrix}.
\]

Consequently, the hedging demands are given by
\[
\pi^H(t) = \begin{bmatrix} \pi^{1,H}(t) \\ \pi^{2,H}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1 - \frac{1}{\gamma}}{\gamma} + \frac{B(t) \tilde{X}}{X(t)} \end{bmatrix}.
\]

Again as it was in the model with CRRA utility both market prices of risk are constant and so the hedging is needed exclusively against the interest rate risk and is performed solely by the bond. In contrast to CRRA portfolios the interest rate hedging for HARA utility with the subsistence level is combined with the liability hedging to ensure the subsistence level.

Analytical formulas for optimal portfolios in the model with HARA utility and CIR [1985] interest rate dynamics are not available yet to the best of our knowledge. It is our task to simulate them in the subsequent sections of this chapter.

1.5.2 Malliavin calculus

Instead of solving for optimal portfolios as done in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] we can simulate the mean-variance and hedging demands according to the approach of Detemple et al. [2003]. In the models of financial markets as described in section 2.4 the simulation of hedging portfolios is done according to the following guidelines. First, the simulation of Malliavin derivatives of market price of risk and interest rate is performed. The Malliavin derivative of the stock market price of risk equals zero, \( D_1 \lambda_1 = D_2 \lambda_1 = 0 \), because the stock market price of risk is constant, \( \lambda_1 = \lambda \). The bond market price of risk is a function of interest rate, \( \lambda_2 = \lambda r(t)^\gamma \). Because the interest rate, \( r(t) \), is driven only by the Brownian motion \( z_r(t) \) the bond market price of risk is insensitive to the perturbations in the path of \( z(t) \) and consequently the Malliavin derivative of \( \lambda_2 \) with respect to \( z(t) \) equals zero, \( D_1 \lambda_2 = 0 \).

The dependence of bond market price of risk on the interest rate implies in turn from the chain rule of Malliavin calculus that \( D_2 \lambda_2 = \partial_2 \lambda_2(s, V(s)) D_2 r_s \). Given the kind of relation between \( \lambda_2 \) and \( r(t) \) we end up with \( D_2 \lambda_2 = \lambda_2 \eta (r(t))^{\gamma - 1} D_2 r_s \). Consequently, even though the bond market price of risk is not constant it is still a function of interest rate and hence the only hedging is
needed against the interest rate risk. The dynamics of the Malliavin derivative of the interest rate in our models of the financial market are from diffusion (2.16) with

\[ \mu^V = a(b - r(t)), \]
\[ \sigma^V = -\sigma_r (r(t))^\eta, \]
given by

\[ dD_{2t}r_s = -aD_{2t}r_s ds - \sigma_r \eta (r(t))^{\eta-1} D_{2t}r_s dz_r, \]

with the initial condition \( \lim_{s \to t} D_{2t}r_s = -\sigma_r (r(t))^\eta \). Summarizing, all the above implies that when the utility belongs to HARA and financial market models are as in section 2.4 the interest rate risk hedging portfolio can be written as

\[ \pi^H(t) = -\frac{(\sigma(\cdot)^T)^{-1}}{\gamma X_t} E_t \left( \zeta_{t,T} \left[ X_T (\gamma - 1) + \hat{X} \right] MDE^T \right), \]

(2.24)

where the Malliavin derivative expression (MDE)

\[ MDE \equiv \int_t^T [0, D_{2t}r_s] ds + \int_t^T (dz_r + \lambda_r ds)^T \begin{bmatrix} 0 \\ \lambda_r \eta (r(t))^{\eta-1} \end{bmatrix} [0, D_{2t}r_s] \]

can be rewritten as

\[ MDE^T \equiv \begin{bmatrix} \int_t^T D_{2t}r_s ds + \int_t^T (dz_r + \lambda_r (r(t))^\eta ds) \lambda_r \eta (r(t))^{\eta-1} D_{2t}r_s \end{bmatrix}. \]

(2.25)

Adequately, for CRRA utility with the subsistence level the hedging demand is given by

\[ \pi^H(t) = -\frac{\gamma}{\gamma - 1} \left( \sigma(\cdot)^T \right)^{-1} E_t \left( \zeta_{t,T} MDE^T \right). \]

(2.26)

### 1.5.3 From Malliavin calculus to analytical formulas

In this section we show how the analytical formulas from section 2.5.1 in the model of Bajeux-Besnainou et al. [2003] are derived using the Malliavin calculus approach from section 2.3.2 and 2.5.2. We start with the derivation for CRRA utility.

In the model of Bajeux-Besnainou et al. [2003] the Malliavin derivative of short term interest rate satisfies the differential equation

\[ dD_{2t}r_s = -aD_{2t}r_s ds, \]
with the initial condition \( \lim_{t \to -1} D_{2t} r_s = -\sigma_r \). The solution to this equation which has the integral form

\[
D_{2t} r_s = -\int_t^s a D_{2t} r_v dv,
\]

is given by

\[
D_{2t} r_s = -\sigma_r e^{-a(s-t)}.
\]

Hence, the integral \( \int_t^T D_{2t} r_s ds \) has the form

\[
\int_t^T D_{2t} r_s ds = -\sigma_r \int_t^T e^{-a(s-t)} ds = \frac{\sigma_r}{a} \left( e^{-a(T-t)} - 1 \right).
\]

Hence,

\[
\int_t^T D_{2t} r_s ds = -\frac{\sigma_r}{a} (1 - e^{-a(T-t)}),
\]

which after the comparison with formula (2.22) gives

\[
\int_t^T D_{2t} r_s ds = -\sigma_T(t). \tag{2.27}
\]

Because the market prices of risk are constant in the model of Bajeux-Besnainou et al. [2003] the vector \( \partial_\lambda \) is the vector of zeros. Substituting (2.27) into formula (2.25) we end up with

\[
MDE^T = \begin{bmatrix} 0, \int_t^T D_{2t} r_s \end{bmatrix}^T = [0, -\sigma_T(t)]^T.
\]

Hence, the formula for the interest rate risk hedging component takes the form

\[
\pi^H(t) = -\frac{\gamma}{\gamma - 1} \left( \sigma(\cdot)^T \right)^{-1} \begin{bmatrix} 0 \\ -\sigma_T(t) \end{bmatrix}.
\]

Inverting \( \sigma^T \) and substituting into the above formula yields

\[
\pi^H(t) = \frac{\gamma}{\gamma - 1} \frac{1}{\sigma_1 \sigma_T(t)} \begin{bmatrix} \sigma_T(t) & 0 \\ -\sigma_2 & \sigma_1 \end{bmatrix} \begin{bmatrix} 0 \\ -\sigma_T(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\gamma}{\gamma - 1} \end{bmatrix}.
\]

This is the same as the expression for the interest rate risk hedging component for CRRA utility from section 2.5.1.

We showed that when the utility is HARA function with the subsistence level \( \bar{X} \), the relative risk risk aversion coefficient is given by (2.17) and the optimal terminal wealth by (2.18). For such a form of the utility function the subsistence level is guaranteed and hence \( 1_{X_T > 0} = 1 \). Substituting for the relative risk aversion coefficient and \( 1_{X_T > 0} = 1 \) we obtain

\[
a(\cdot)^T \equiv E_t \left( \zeta_{t,T} \frac{X_T}{X_t} \left( 1 - \left( \frac{\gamma X_T}{X_T - \bar{X}} \right)^{-1} \right) \int_t^T D_{1t} r_s ds \right),
\]

\[
b(\cdot)^T \equiv [0, 0],
\]

\[
c(\cdot) \equiv E_t \left( \zeta_{t,T} \frac{X_T}{X_t} \left( \frac{\gamma X_T}{X_T - \bar{X}} \right)^{-1} \right).
\]
After some simplifications we get
\[
T = \begin{cases} 0, & -\sigma_T(t) \left( 1 - \frac{1}{\gamma} \right) E_t \left( \zeta_{t,T} \frac{X_T}{X_t} \right) - \frac{\sigma_T(t)}{\gamma X_t} E_t \left( \zeta_{t,T} \right) \end{cases},
\]
\[
b(T) = [0, 0],
\]
\[
c(T) = E_t \left( \zeta_{t,T} \left( \frac{X_T - \tilde{X}}{X_t - \tilde{X}} \right) \right).
\]

From the budget constraint \( E_t \left( \zeta_{t,T} \frac{X_T}{X_t} \right) = 1 \) and from the risk-neutral valuation principle \( E_t \left( \zeta_{t,T} \right) = B_T(t) \). Hence,
\[
a(T) = \begin{cases} 0, & -\sigma_T(t) \left( 1 - \frac{1}{\gamma} \right) - \frac{\sigma_T(t)}{\gamma X_t} B_T(t) \end{cases}.
\]

As a result we have
\[
\pi^{MV}(t) = \left( \sigma^T \right)^{-1} \frac{\lambda}{R(X_t)} c(t, \lambda, r) = \left[ \frac{\lambda \sigma_T(t)}{-\lambda \sigma_T + \lambda, \sigma_1} \right] \left( \frac{1}{\gamma} - \frac{\hat{X} B_T(t)}{\gamma X_t} \right)
\]
and
\[
\pi^H(t) = -\frac{1}{\sigma_1 \sigma_T(t)} \left[ \begin{array}{cc} \sigma_T(t) & 0 \\ -\sigma_2 & \sigma_1 \end{array} \right] \left[ \begin{array}{cc} -\sigma_T(t) \left( 1 - \frac{1}{\gamma} \right) - \frac{\sigma_T(t)}{\gamma X_t} B_T(t) \end{array} \right] = \left[ \begin{array}{cc} 0 & 0 \\ 1 - \frac{1}{\gamma} + \frac{\hat{X} B_T(t)}{\gamma X_t} \end{array} \right].
\]

This is the same as the expression for the interest rate risk hedging component for HARA utility from section 2.5.1.

1.6 Simulation experiment

In the experiment which results are presented in the next section we assume \( T = 1 \) and \( R(X_t) = R(X_T) = 1 - \gamma = 2 \) for CRRA utility. For HARA utility \( T = 1 \) and \( R(X_t) \) and \( R(X_T) \) are given in (2.17). In the part of simulation experiment referring to HARA utility we also assume the subsistence level of 1000 and initial wealth of 1100 (HARA 1), 1300 (HARA 2) and 1500 (HARA 3). The values of remaining parameters describing financial markets from section 2.4 are given in Table 2.1. We choose the estimates for the interest rate processes from Chan et al. [1992]. The initial rate, \( r \), is set at the mean of the sample used therein. As far as market prices of risk and volatilities of assets in the model with \( \eta = 0 \) are concerned we follow Boulier et al. [2001]. These values imply 1-year-bond volatility of 1.83% and 1-year-bond risk premium of 0.28% at the initial date, stock volatility of 20.00% and stock risk premium of 6.00%. The correlation between stock and bond markets measured as \( \rho_{S,B} \equiv \sigma_2 \sigma_S^{-1} \) with
\[ \sigma_S \equiv (\sigma_1^2 + \sigma_2^2)^{0.5} \] equals 30.00\%. For the model with \( \eta = 0.5 \) we choose \( \sigma_1 \) and \( \sigma_2 \) such that \( \rho_{S, r} = \sigma_2 \sigma_S^{-1} \sqrt{T} \) and \( \sigma_S \equiv (\sigma_1^2 + \sigma_2^2)^{0.5} \) are as in \( \eta = 0 \) model.

In turn, \( \lambda_r \) makes the initial prices of bonds with maturities \( T = 1, 2, ..., 10 \) years differ as little as possible in terms of mean squared error from the prices in the \( \eta = 0 \) model. Then, we choose \( \lambda \) such that the stock risk premium at initial date is as in the \( \eta = 0 \) model. These values imply 1-year-bond volatility of 2.15\% and 1-year-bond premium of 0.15\% at the initial date, stock volatility of 20\% and stock premium of 6\%.

### Table 2.1. Parameters of the financial market

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<th>CIR</th>
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<td>6.00</td>
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<tr>
<td>( \sigma^2 )</td>
<td>2.00</td>
<td>8.54</td>
<td>( \sigma^2 )</td>
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<tr>
<td>( \lambda )</td>
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<td>6.715</td>
<td>( \lambda )</td>
<td>15.28</td>
<td>26.18</td>
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</tr>
</tbody>
</table>

The numbers in Table 2.1 stand for \( \dagger \) speed of interest rate reversion, \( \dagger \) mean to which the interest rate reverts, \( \dagger \) volatility of interest rate process, \( \dagger \) initial value of interest rate, \( \dagger \) volatilities of stock price process, \( \dagger \) market prices of risk

### 1.7 Results

Using the formulas from section 2.5.1 we calculate the demand for risky assets in total and hedging continuous-time portfolios at the initial date for two models of the financial market described in 2.4 and for both CRRA and HARA utility except when the utility belongs to HARA and the interest rate dynamics is given by the CIR [1985] model. In this case the optimal continuous-time policy is not known to the best of our knowledge. These results are reported in the upper panels of Tables 2.2-2.5. Discretizing the continuous-time processes and using the formulas in 2.5.2 we simulate adequate portfolios by considering \( 3^n \) paths, where \( n = 2, 3, ..., 11 \). The time step \( \frac{1}{n} \) as well as the number of paths are chosen to make the simulated results as much comparable as possible with the outcome from the tree model we refer to in the next chapter. Simulated portfolios are reported in the lower panels of Tables 2.2-2.5. The outcome for CRRA is displayed in Tables 2.2. The results for HARA are given in 2.3-2.5.

Numbers from Table 2.2 reveal first that the total demand for bond is above the continuous-time counterpart in both models of financial markets. This difference is however much greater for \( \eta = 0.5 \) than \( \eta = 0 \) model. The surplus in relative terms diminishes in the simulated portfolios from 351.94\% to 2.56\% and from 61.54\% to 00.16\% adequately in \( \eta = 0.5 \) and \( \eta = 0 \) models. Second, as far as the hedging demand for bond is concerned it is still overestimated in
the simulated portfolios. The surplus of simulated hedging demand for bond reduces in adequate models $\eta = 0.5$ and $\eta = 0$ from 259.10% to 1.88% and from 293.38% to 0.75%. In general the convergence is reasonable.

Table 2.2. Continuous-time and simulated total ($\pi^1, \pi^2$) and hedging ($\pi^{1,H}, \pi^{2,H}$) portfolios for CRRA

<table>
<thead>
<tr>
<th>Continuous-time portfolios</th>
<th></th>
<th></th>
<th></th>
<th>Simulated portfolios</th>
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<th></th>
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</thead>
<tbody>
<tr>
<td></td>
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<td>$\eta = 0.5$</td>
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<td>$\pi^{2,H}$</td>
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<td>$\pi^2$</td>
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Numbers in Table 2.2 are the fractions of risky assets in total, $\pi^k$, and hedging, $\pi^{k,H}$, $k = 1, 2$, optimal CRRA portfolios at the initial date for two models of the financial market respectively with Vasicek ($\eta = 0$) and CIR ($\eta = 0.5$) dynamics.

As far as the results for HARA utility are concerned the following conclusions can be formulated. First, the convergence of simulated portfolios is reasonable in the model with Vasicek [1977] interest rate. Second, for the model with CIR [1985] interest rate dynamics the fraction of wealth invested in stock equals around 9% for HARA1, 17% for HARA 2 and 23% for HARA3. The fraction of wealth invested in bond equals around 24% for HARA1, 20% for HARA2 and 17-18% for HARA3. Hence, the higher the ratio of initial wealth to the subsistence level the less fraction of wealth invested in bond. This means that in the institutions with certain liability to be met the importance of hedging against interest rate risk decreases with the growing financial security of the institution. As in the case of CRRA the share of stock does not contain the hedging components. Hedging demand for bond is higher than total demand implying that in the mean-variance portfolio short position is hold in bond.
Table 2.3. Simulated total ($\pi^1, \pi^2$) and hedging ($\pi^{1,H}, \pi^{2,H}$) portfolios for HARA1

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Simulated portfolios

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Table 2.4. Simulated total ($\pi^1, \pi^2$) and hedging ($\pi^{1,H}, \pi^{2,H}$) portfolios for HARA2

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Simulated portfolios

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Table 2.5. Simulated total ($\pi^1, \pi^2$) and hedging ($\pi^{1,H}, \pi^{2,H}$) portfolios for HARA3

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<td>$\pi^2$</td>
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</table>

Numbers in Tables 2.3–2.5 are the fractions of risky assets in total, $\pi^k$, and hedging, $\pi^{k,H}$, $k = 1, 2$, optimal HARA portfolios with initial wealth of 1100, 1300 and 1500 adequately at the initial date for two models of the financial market respectively with Vasicek ($\eta = 0$) and CIR ($\eta = 0.5$) dynamics.

1.8 Conclusions

In this chapter we introduced the concept of interest rate risk hedging in the optimal portfolio choice. We investigated the convergence of simulated CRRA optimal portfolios and their interest rate risk hedging components to the continuous-time limits in the models with Vasicek [1977] and CIR [1985] interest rate dynamics. We note that both total and hedging demand for bond is overestimated in the simulated portfolios in comparison with the continuous-time demand. We also simulated HARA optimal portfolios and spotted the convergence of these portfolios to the continuous-time equivalents in the model with Vasicek [1977] interest rate dynamics. As far as the model with HARA utility and CIR [1985] interest rate is concerned the analytical formulas for optimal portfolios are not available to the best of our knowledge. Hence, their simulation can give us an indication about investors’ optimal policy. We find that with an increase of the ratio of initial wealth to the subsistence level the stock investment is increased, while the bond investment diminishes in both total and hedging portfolios. Hence, with an increase of the safety buffer more is invested in the most risky asset and less interest rate risk hedging is performed. Regarding HARA
utility and Vasicek [1977] interest rate dynamics we also showed how to obtain the analytical formulas for optimal portfolios using the Malliavin calculus.